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THE EXACT SOLUTIONS FOR THE NATURAL FREQUENCIES AND MODE SHAPES OF NON-UNIFORM BEAMS WITH MULTIPLE SPRING–MASS SYSTEMS

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For a beam carrying n spring–mass systems, if the left side and right side of each attaching point and each end of the beam are regarded as nodes, then considering the compatibility of deformations and the equilibrium of forces between the two adjacent beam segments at each attaching point and incorporating with the equation of motion for each spring–mass system, simultaneous equations may be obtained for the vth attaching point, where the unknowns for the simultaneous equations are composed of the integration constants for the eigenfunctions of the vth and $(v+1)$th beam segments and the associated modal displacements of the vth sprung mass. It is evident that if these unknowns are considered as the nodal displacements, then the coefficient matrix of the simultaneous equations will be equivalent to the element stiffness matrix for the vth attaching point (associated with the vth and $(v+1)$th beam segments). In view of the last fact, one may use the numerical assembly method (NAM) for the conventional finite element method to obtain the overall simultaneous equations for the overall (n) attaching points (associated with the overall $(n+1)$ beam segments) by taking into account the boundary conditions of the whole beam. The solutions for the coefficient determinant of the overall simultaneous equations to be equal to zero will give the "exact" natural frequencies of the constrained beam (carrying multiple (n) spring–mass systems) and the substitution of each corresponding values of the integration constants into the associated eigenfunctions for each attaching point will determine the corresponding mode shapes. Since no discretization on the continuous beam was made in the present approach (NAM), the natural frequencies and the corresponding mode shapes obtained are the exact ones.

1. INTRODUCTION

For a uniform beam carrying various concentrated elements (such as rigidly attached point masses, elastically mounted lumped masses, linear springs and/or rotational springs), the free vibration problem has been studied by a lot of researchers [1–12]. However, for a non-uniform beam, even without any attachments, the researches on their dynamic behaviors are relatively fewer [13–18]. As to the free vibration analysis of the non-uniform beams carrying multiple concentrated elements [19], the information concerned is rare and this is one of the reasons why the problem in this aspect is studied.

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In reference [10], it has been found that the eigenequation for a uniform Euler–Bernoulli beam carrying multiple sprung masses takes the form $[B]\{C\} = 0$. Since the order of the overall coefficient matrix $[B]$ is $5n + 4$, where n is the total number of the sprung masses, the order of $[B]$ is 9 for the beam carrying one sprung mass and 14 for the beam carrying two sprung masses. It is evident that the explicit expression for the eigen equation $[B]\{C\} = 0$ will become lengthy and intractable for the cases with $n > 2$, hence the literature relating to the free vibration analysis of a non-uniform beam carrying more than two concentrated attachments is rare [19]. However, the numerical assembly method (NAM) presented in that paper may easily tackle the cases with any number of concentrated attachments. In reference [12], the NAM was used to perform the free vibration analysis of a uniform Timoshenko beam carrying multiple spring–mass systems and satisfactory results were achieved. To the authors' knowledge, besides the conventional finite element method (FEM), no other effective techniques were presented to solve the title problem, particularly the ones that may provide the exact solutions. For this reason, this paper tries to introduce the NAM to solve the problem.

To realize the effectiveness of the NAM, the lowest five natural frequencies and some of the corresponding mode shapes of a double-tapered beam carrying one, three and five spring–mass systems were calculated respectively. In each case, six boundary conditions were studied: free–clamped, clamped–free, simply supported–clamped, clamped–simply supported, clamped–clamped, and simply supported–simply supported. It has been found that the agreement between the NAM results and the FEM results is excellent.

2. EIGENFUNCTIONS FOR THE CONSTRAINED NON-UNIFORM BEAM

Figure 1 shows a non-uniform cantilever beam carrying n spring–mass systems. The whole beam with length L is subdivided into $(n + 1)$ segments by the attaching point v located at $x = x_v$ $(v = 1, 2, \ldots, n)$, where (○) denotes the vth attaching point and ( ) denotes the vth beam segment. In addition, the left end and the right end of the beam are denoted by Ⓛ and Ⓡ, respectively.

The equation of motion for a bare non-uniform beam is given by [18, 20]

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\,\frac{\partial^2 y(x,t)}{\partial x^2}\right] + \rho A(x)\,\frac{\partial^2 y(x,t)}{\partial t^2} = 0, \tag{1}$$

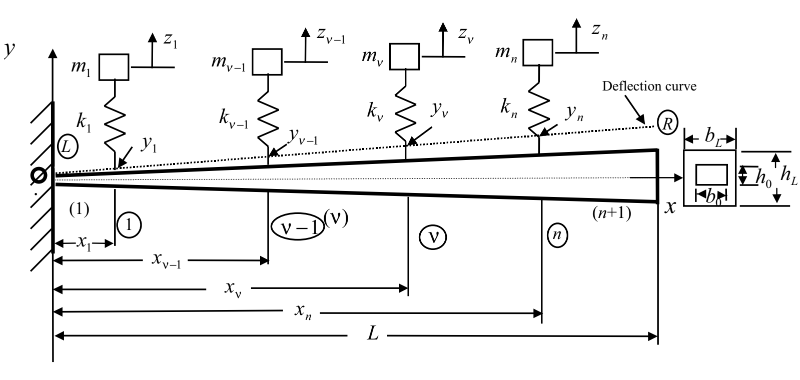

Figure 1. A non-uniform cantilever beam carrying n spring–mass systems.

where $y(x, t)$ is the transverse deflection, E is Young's modulus, $A(x)$ is the cross-sectional area at the position x, $I(x)$ is the moment of inertia of $A(x)$, ρ is the mass density of the beam material and t is time.

If $A(x)$ and $I(x)$ are given by

$$A(x) = A_0\left[(\alpha - 1)\frac{x}{L} + 1\right]^2, \qquad I(x) = I_0\left[(\alpha - 1)\frac{x}{L} + 1\right]^4 \tag{2}$$

and free vibration of the beam takes the form

$$y(x, t) = \bar{Y}(x)\mathrm{e}^{\mathrm{i}\bar{\omega}t}, \tag{3}$$

then the substitution of equations (2) and (3) into equation (1) yields

$$\left[(\alpha - 1)\frac{x}{L} + 1\right]^4 \frac{\mathrm{d}^4\bar{Y}(x)}{\mathrm{d}x^4} + 8\left[(\alpha - 1)\frac{x}{L} + 1\right]^3\left(\frac{\alpha - 1}{L}\right)\frac{\mathrm{d}^3\bar{Y}(x)}{\mathrm{d}x^3}$$
$$+ 12\left[(\alpha - 1)\frac{x}{L} + 1\right]^2\left(\frac{\alpha - 1}{L}\right)^2\frac{\mathrm{d}^2\bar{Y}(x)}{\mathrm{d}x^2} - \frac{\rho A_0\bar{\omega}^2}{EI_0}\left[(\alpha - 1)\frac{x}{L} + 1\right]^2\bar{Y}(x) = 0. \tag{4}$$

In equation (2), A_0, I_0, b_0 and h_0 are the cross-sectional area, moment of inertia, width and height of the cross-section at $x = 0$, b_L and h_L are the width and height of the cross-section at $x = L$, respectively, and $\alpha = b_L/b_0 = h_L/h_0$ is the taper ratio of the beam. In equation (3), $\bar{\omega}$ is the natural frequency of the constrained non-uniform beam and $\bar{Y}(x)$ is the amplitude of $y(x, t)$.

Introducing the non-dimensional coefficient

$$\xi = (\alpha - 1)\frac{x}{L} + 1 \tag{5}$$

will render equation (4) to

$$\xi^4 Y''''(\xi) + 8\xi^3 Y'''(\xi) + 12\xi^2 Y''(\xi) - \xi^2\left[\frac{L\Omega}{(\alpha - 1)}\right]^4 Y(\xi) = 0, \tag{6}$$

where

$$(\Omega L)^4 = \frac{\rho A_0\bar{\omega}^2 L^4}{EI_0}.$$

It is noted that $\xi = 1$ at $x = 0$ and $\xi = \alpha$ at $x = L$, as one may see from equation (5).

The general solution of equation (6) is given by [20, 21]

$$\bar{Y}(\xi) = \xi^{-1}[C_1 J_2(\beta\sqrt{\xi}) + C_2 Y_2(\beta\sqrt{\xi}) + C_3 I_2(\beta\sqrt{\xi}) + C_4 K_2(\beta\sqrt{\xi})] \tag{7a}$$

with

$$\beta = 2L\Omega/(\alpha - 1), \tag{7b}$$

where C_i ($i = 1$–4) are the integration constants, J_2 and Y_2 are the second order Bessel function of first and second kinds, while I_2 and K_2 is the second order modified Bessel function of first and second kinds.

Equation (7a) represents the eigenfunction for the transverse deflection of the constrained beam. Once the natural frequencies $\bar{\omega}_j$ ($j = 1, 2, \dots$) and the constants for each attaching point, C_i ($i = 1$–4), are determined from the next sections, one may obtain the values of $\bar{Y}_j(\xi)$. The latter are the mode shapes of the constrained beam corresponding to the natural frequency $\bar{\omega}_j$.

For the vth beam segment, from equation (7a) one has

$$\bar{Y}_v(\xi_v) = \xi_v^{-1}[C_{v1}J_2(\beta\sqrt{\xi_v}) + C_{v2}Y_2(\beta\sqrt{\xi_v}) + C_{v3}I_2(\beta\sqrt{\xi_v}) + C_{v4}K_2(\beta\sqrt{\xi_v})], \quad (8a)$$

where

$$\xi_v = (\alpha - 1)\frac{x_v}{L} + 1, \quad (8b)$$

The differentiation of $\bar{Y}_v(\xi_v)$ with respect to ξ_v yields

$$\bar{Y}'_v(\xi_v) = -\frac{\beta}{2}\xi_v^{-3/2}[C_{v1}J_3(\beta\sqrt{\xi_v}) + C_{v2}Y_3(\beta\sqrt{\xi_v}) - C_{v3}I_3(\beta\sqrt{\xi_v}) + C_{v4}K_3(\beta\sqrt{\xi_v})], \quad (9)$$

$$\bar{Y}''_v(\xi_v) = \left[\frac{\beta}{2}\right]^2 \xi_v^{-2}[C_{v1}J_4(\beta\sqrt{\xi_v}) + C_{v2}Y_4(\beta\sqrt{\xi_v}) + C_{v3}I_4(\beta\sqrt{\xi_v}) + C_{v4}K_4(\beta\sqrt{\xi_v})], \quad (10)$$

$$\bar{Y}'''_v(\xi_v) = -\left[\frac{\beta}{2}\right]^3 \xi_v^{-5/2}[C_{v1}J_5(\beta\sqrt{\xi_v}) + C_{v2}Y_5(\beta\sqrt{\xi_v}) - C_{v3}I_5(\beta\sqrt{\xi_v}) + C_{v4}K_5(\beta\sqrt{\xi_v})]. \quad (11)$$

3. COEFFICIENT MATRIX $[B_v]$ FOR THE vTH ATTACHING POINT

Compatibility for the deformations at the attaching point requires that

$$\bar{Y}_v^L(\xi_v) = \bar{Y}_v^R(\xi_v), \quad \bar{Y}_v'^L(\xi_v) = \bar{Y}_v'^R(\xi_v), \quad \bar{Y}_v''^L(\xi_v) = \bar{Y}_v''^R(\xi_v). \quad (12a\text{–}c)$$

For the force equilibrium at the attaching point, one has

$$\begin{aligned} &4(\alpha-1)^3\xi^3\bar{Y}_v''^L(\xi_v) + (\alpha-1)^3\xi^4\bar{Y}_v'''^L(\xi_v) + F_s \\ &\quad = 4(\alpha-1)^3\xi^3\bar{Y}_v''^R(\xi_v) + (\alpha-1)^3\xi^4\bar{Y}_v'''^R(\xi_v), \end{aligned} \quad (13)$$

where F_s is the interactive force between the beam and the attached spring–mass system and is given by [10, 23]

$$F_s = \frac{m_v\bar{\omega}^2}{1 - m_v\bar{\omega}^2/k_v}\cdot\bar{Y}_v^L(x_v) = \frac{m_v^*[\frac{1}{3}(\alpha-1)^2 + (\alpha-1) + 1](\Omega L)^4}{1 - m_v^*[\frac{1}{3}(\alpha-1)^2 + (\alpha-1) + 1](\Omega L)^4/k_v^*}\cdot\bar{Y}_v^L(\xi_v). \quad (14)$$

The equation of motion for the vth sprung mass is given by

$$m_v\ddot{z}_v + k_v(z_v - y_v) = 0, \quad (15)$$

where m_v and k_v are the lumped mass and spring constant of the vth spring–mass system, respectively, $\ddot{z}_v$ and z_v are the acceleration and displacement of the vth sprung mass and y_v is the transverse deflection of the beam at the vth attaching point.

Similar to equation (3), free vibration of the νth sprung mass takes the form

$$z_v = Z_v(\xi)e^{i\bar{\omega}t}, \qquad v = 1, 2, \ldots, n \tag{16}$$

where $Z_v(\xi)$ is the amplitude (or modal displacement) of the sprung mass.

Insertion of equations (3) and (16) into equation (15) gives

$$\bar{Y}_v + (\gamma_v^2 - 1)Z_v = 0 \tag{17a}$$

or

$$\bar{Y}_v + \left\{\frac{m_v^*[\frac{1}{3}(\alpha-1)^2 + (\alpha-1) + 1]}{k_v^*}(\Omega L)^4 - 1\right\}Z_v = 0, \tag{17b}$$

where

$$\gamma_v^2 = \frac{\bar{\omega}^2}{\omega_v^2} = \frac{(\Omega L)^4 EI_0}{L^4 \rho A_0}\cdot\frac{m_v}{k_v} = \frac{m_v^*[\frac{1}{3}(\alpha-1)^2 + (\alpha-1) + 1]}{k_v^*}\cdot(\Omega L)^4, \tag{18}$$

$$\omega_v = \sqrt{k_v/m_v}, \qquad m_v^* = \frac{m_v}{m_b} = \frac{m_v}{\rho A_0 L[\frac{1}{3}(\alpha-1)^2 + (\alpha-1) + 1]}, \tag{19, 20}$$

$$k_v^* = \frac{k_v}{(EI_0/L^3)}, \tag{21}$$

where $m_b = \rho A_0 L[\frac{1}{3}(\alpha-1)^2 + (\alpha-1) + 1]$ is the beam mass.

The substitution of equations (8)–(11) into equations (12), (13) and (17) leads to

$$C_{v1}J_2(\beta\sqrt{\xi_v}) + C_{v2}Y_2(\beta\sqrt{\xi_v}) + C_{v3}I_2(\beta\sqrt{\xi_v}) + C_{v4}K_2(\beta\sqrt{\xi_v})$$
$$- C_{v+1,1}J_2(\beta\sqrt{\xi_v}) - C_{v+1,2}Y_2(\beta\sqrt{\xi_v}) - C_{v+1,3}I_2(\beta\sqrt{\xi_v}) - C_{v+1,4}K_2(\beta\sqrt{\xi_v}) = 0, \tag{22a}$$

$$C_{v1}J_3(\beta\sqrt{\xi_v}) + C_{v2}Y_3(\beta\sqrt{\xi_v}) - C_{v3}I_3(\beta\sqrt{\xi_v}) + C_{v4}K_3(\beta\sqrt{\xi_v}) - C_{v+1,1}J_3(\beta\sqrt{\xi_v})$$
$$- C_{v+1,2}Y_3(\beta\sqrt{\xi}) + C_{v+1,3}I_3(\beta\sqrt{\xi_v}) - C_{v+1,4}K_3(\beta\sqrt{\xi_v}) = 0, \tag{22b}$$

$$C_{v1}J_4(\beta\sqrt{\xi_v}) + C_{v2}Y_4(\beta\sqrt{\xi_v}) + C_{v3}I_4(\beta\sqrt{\xi_v}) + C_{v4}K_4(\beta\sqrt{\xi_v})$$
$$- C_{v+1,1}J_4(\beta\sqrt{\xi_v}) - C_{v+1,2}Y_4(\beta\sqrt{\xi_v}) - C_{v+1,3}I_4(\beta\sqrt{\xi_v}) - C_{v+1,4}K_4(\beta\sqrt{\xi_v}) = 0, \tag{22c}$$

$$8\beta^2[C_{v1}J_4(\beta\sqrt{\xi_v}) + C_{v2}Y_4(\beta\sqrt{\xi_v}) + C_{v3}I_4(\beta\sqrt{\xi_v}) + C_{v4}K_4(\beta\sqrt{\xi_v})]$$
$$- \beta^3\xi_v^{1/2}[C_{v1}J_5(\beta\sqrt{\xi_v}) + C_{v2}Y_5(\beta\sqrt{\xi_v}) - C_{v3}I_5(\beta\sqrt{\xi_v}) + C_{v4}K_5(\beta\sqrt{\xi_v})]$$

$$+ 8\theta_v \xi_v^{-2}[C_{v1}J_2(\beta\sqrt{\xi_v}) + C_{v2}Y_2(\beta\sqrt{\xi_v}) + C_{v3}I_2(\beta\sqrt{\xi_v}) + C_{v4}K_2(\beta\sqrt{\xi_v})]$$

$$- 8\beta^2[C_{v+1,1}J_4(\beta\sqrt{\xi_v}) + C_{v+1,2}Y_4(\beta\sqrt{\xi_v}) + C_{v+1,3}I_4(\beta\sqrt{\xi_v}) + C_{v+1,4}K_4(\beta\sqrt{\xi_v})]$$

$$+ \beta^3\xi_v^{1/2}[C_{v+1,1}J_5(\beta\sqrt{\xi_v}) + C_{v+1,2}Y_5(\beta\sqrt{\xi_v}) - C_{v+1,3}I_5(\beta\sqrt{\xi_v}) + C_{v+1,4}K_5(\beta\sqrt{\xi_v})] = 0, \tag{22d}$$

$$\xi_v^{-1}[C_{v1}J_2(\beta\sqrt{\xi_v}) + C_{v2}Y_2(\beta\sqrt{\xi_v}) + C_{v3}I_2(\beta\sqrt{\xi_v}) + C_{v4}K_2(\beta\sqrt{\xi_v})] + (\gamma_v^2 - 1)Z_v = 0, \tag{22e}$$

where

$$\theta_v = \left\{\frac{m_v^*[\frac{1}{3}(\alpha-1)^2 + (\alpha-1) + 1](\Omega L)^4}{1 - (m_v^*[\frac{1}{3}(\alpha-1)^2 + (\alpha-1) + 1](\Omega L)^4)/k_v^*} \frac{1}{(\alpha-1)^3}\right\},$$

$$\gamma_v^2 = \frac{m_v^*[\frac{1}{3}(\alpha-1)^2 + (\alpha-1) + 1]}{k_v^*} \cdot (\Omega L)^4. \tag{22f}$$

It is noted that, in equations (12) and (13), the "left side" of the vth attaching point located at $x = x_v$ belongs to the segment (v) and the "right side" belongs to the segment ($v + 1$), thus the associated coefficients are represented by C_{vi} and $C_{v+1,i}$ ($i = 1$–4), respectively, as may be seen from equations (22a)–(22e).

Equations (22a)–(22e) can be written in matrix form as

$$[B_v]\{C_v\} = 0, \tag{23}$$

where

$$\{C_v\} = \{C_{v1}\ C_{v2}\ C_{v3}\ C_{v4}\ C_{v+1,1}\ C_{v+1,2}\ C_{v+1,3}\ C_{v+1,4}\ Z_v\}$$

$$= \{\bar{C}_{4v-3}\ \bar{C}_{4v-2}\ \bar{C}_{4v-1}\ \bar{C}_{4v}\ \bar{C}_{4v+1}\ \bar{C}_{4v+2}\ \bar{C}_{4v+3}\ \bar{C}_{4v+4}\ \bar{C}_{4v+5}\}, \tag{24a}$$

$$\bar{C}_{4v-3} = C_{v1},\ \bar{C}_{4v-2} = C_{v2}, \ldots,\ \bar{C}_{4v+4} = C_{v+1,4},\ \bar{C}_{4v+5} = Z_v \tag{24b}$$

and

$$\begin{array}{c} \\ \\ [B_v] = \end{array} \begin{array}{c} \begin{array}{ccccccccc} 4v-3 & 4v-2 & 4v-1 & 4v & 4v+1 & 4v+2 & 4v+3 & 4v+4 & 4v+5 \end{array} \\ \left[\begin{array}{ccccccccc} J_2(\delta_v) & Y_2(\delta_v) & I_2(\delta_v) & K_2(\delta_v) & -J_2(\delta_v) & -Y_2(\delta_v) & -I_2(\delta_v) & -K_2(\delta_v) & 0 \\ J_3(\delta_v) & Y_3(\delta_v) & -I_3(\delta_v) & K_3(\delta_v) & -J_3(\delta_v) & -Y_3(\delta_v) & I_3(\delta_v) & -K_3(\delta_v) & 0 \\ J_4(\delta_v) & Y_4(\delta_v) & I_4(\delta_v) & K_4(\delta_v) & -J_4(\delta_v) & -Y_4(\delta_v) & -I_4(\delta_v) & -K_4(\delta_v) & 0 \\ \Delta_{1v} & \Delta_{2v} & \Delta_{3v} & \Delta_{4v} & -\Delta_{5v} & -\Delta_{6v} & -\Delta_{7v} & -\Delta_{8v} & 0 \\ \xi_v^{-1}J_2(\delta_v) & \xi_v^{-1}Y_2(\delta_v) & \xi_v^{-1}I_2(\delta_v) & \xi_v^{-1}K_2(\delta_v) & 0 & 0 & 0 & 0 & -1+\gamma_v^2 \end{array}\right] \end{array} \begin{array}{l} \\ 5v-2 \\ 5v-1 \\ 5v \\ 5v+1 \\ 5v+2 \end{array}, \tag{24c}$$

where

$$\Delta_{1v} = 8\beta^2 J_4(\beta\sqrt{\xi_v}) - \beta^3\xi_v^{1/2} J_5(\beta\sqrt{\xi_v}) + 8\theta_v\xi_v^{-2} J_2(\beta\sqrt{\xi_v}),$$

$$\delta_v = \beta\sqrt{\xi_v}, \quad \Delta_{2v} = 8\beta^2 Y_4(\beta\sqrt{\xi_v}) - \beta^3\xi_v^{1/2} Y_5(\beta\sqrt{\xi_v}) + 8\theta_v\xi_v^{-2} Y_2(\beta\sqrt{\xi_v})$$

$$\Delta_{3v} = 8\beta^2 I_4(\beta\sqrt{\xi_v}) + \beta^3\xi_v^{1/2} I_5(\beta\sqrt{\xi_v}) + 8\theta_v\xi_v^{-2} I_2(\beta\sqrt{\xi_v}),$$

$$\Delta_{4v} = 8\beta^2 K_4(\beta\sqrt{\xi_v}) - \beta^3\xi_v^{1/2} K_5(\beta\sqrt{\xi_v}) + 8\theta_v\xi_v^{-2} K_2(\beta\sqrt{\xi_v}),$$

$$\Delta_{5v} = 8\beta^2 J_4(\beta\sqrt{\xi_v}) - \beta^3\xi_v^{1/2} J_5(\beta\sqrt{\xi_v}),$$

$$\Delta_{6v} = 8\beta^2 Y_4(\beta\sqrt{\xi_v}) - \beta^3\xi_v^{1/2} Y_5(\beta\sqrt{\xi_v}),$$

$$\Delta_{7v} = 8\beta^2 I_4(\beta\sqrt{\xi_v}) + \beta^3\xi_v^{1/2} I_5(\beta\sqrt{\xi_v}),$$

$$\Delta_{8v} = 8\beta^2 K_4(\beta\sqrt{\xi_v}) - \beta^3\xi_v^{1/2} K_5(\beta\sqrt{\xi_v}). \tag{24d}$$

4. COEFFICIENT MATRIX $[B_L]$ FOR THE LEFT END OF THE BEAM

For a cantilever beam with left end clamped, the boundary conditions are

$$\bar{Y}(1) = 0, \quad \bar{Y}'(1) = 0 \tag{25a, b}$$

From Figure 1 one sees that the left end of the beam, Ⓛ, coincides with the left end of the first beam segment ($v = 1$), hence from equations (8), (9), (25a) and (25b) one obtains

$$J_2(\beta)C_{11} + Y_2(\beta)C_{12} + I_2(\beta)C_{13} + K_2(\beta)C_{14} = 0, \tag{26a}$$

$$J_3(\beta)C_{11} + Y_3(\beta)C_{12} - I_3(\beta)C_{13} + K_3(\beta)C_{14} = 0, \tag{26b}$$

The last two expressions can be written in matrix form as

$$[B_L]\{C_L\} = 0, \tag{27}$$

where

$$\begin{matrix} & 1 & 2 & 3 & 4 & \\ [B_L] = & \Big[J_2(\beta) & Y_2(\beta) & I_2(\beta) & K_2(\beta) \Big] & 1 \\ & \Big[J_3(\beta) & Y_3(\beta) & -I_3(\beta) & K_3(\beta) \Big] & 2 \end{matrix}, \tag{28}$$

$$\{C_L\} = \{C_{11} \;\; C_{12} \;\; C_{13} \;\; C_{14}\} = \{\bar{C}_1 \;\; \bar{C}_2 \;\; \bar{C}_3 \;\; \bar{C}_4\}, \tag{29}$$

where the [ ] and { } represent the rectangular matrix and the column vector, respectively, and

$$\bar{C}_1 = C_{11}, \bar{C}_2 = C_{12}, \bar{C}_3 = C_{13}, \bar{C}_4 = C_{14}. \tag{30}$$

In equation (28) and the subsequent equations, the digits shown on the top side and right side of the matrix represent the identification numbers of degrees of freedom (d.o.f.) for the associated constants $\bar{C}_i$ ($i = 1, 2, \ldots$).

5. COEFFICIENT MATRIX $[B_R]$ FOR THE RIGHT END OF THE BEAM

For a cantilever beam with right end free, the boundary conditions are

$$\bar{Y}''(\alpha) = 0, \qquad 4\alpha^{-1}\bar{Y}''(\alpha) + \bar{Y}'''(\alpha) = 0. \tag{31a, b}$$

Since the right end of the beam, Ⓡ, coincides with the right end of the $(n+1)$th segment ($v = n + 1$), as one may see from Figure 1, hence from equations (10), (11), (31a) and (31b) one obtains

$$J_4(\beta\sqrt{\alpha})C_{n+1,1} + Y_4(\beta\sqrt{\alpha})C_{n+1,2} + I_4(\beta\sqrt{\alpha})C_{n+1,3} + K_4(\beta\sqrt{\alpha})C_{n+1,4} = 0, \tag{32a}$$

$$\begin{aligned}&[8J_4(\beta\sqrt{\alpha}) - \beta\alpha^{1/2}J_5(\beta\sqrt{\alpha})]C_{n+1,1} + [8Y_4(\beta\sqrt{\alpha}) - \beta\alpha^{1/2}Y_5(\beta\sqrt{\alpha})]C_{n+1,2}\\ &+ [8I_4(\beta\sqrt{\alpha}) + \beta\alpha^{1/2}I_5(\beta\sqrt{\alpha})]C_{n+1,3} + [8K_4(\beta\sqrt{\alpha}) - \beta\alpha^{1/2}K_5(\beta\sqrt{\alpha})]C_{n+1,4} = 0\end{aligned}, \tag{32b}$$

Equations (32a) and (32b) can be written in matrix form

$$[B_R]\{C_R\} = 0, \tag{33}$$

where

$$\begin{array}{c} \qquad\quad 4n+1 \qquad\quad 4n+2 \qquad\quad 4n+3 \qquad\quad 4n+4 \\ [B_R] = \begin{bmatrix} J_4(\beta\sqrt{\alpha}) & Y_4(\beta\sqrt{\alpha}) & I_4(\beta\sqrt{\alpha}) & K_4(\beta\sqrt{\alpha}) \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \end{bmatrix} \begin{matrix} p-1 \\ p \end{matrix}, \end{array} \tag{34a}$$

$$\begin{aligned}\varepsilon_1 &= [8J_4(\beta\sqrt{\alpha}) - \beta\alpha^{1/2}J_5(\beta\sqrt{\alpha})], \quad \varepsilon_2 = [8Y_4(\beta\sqrt{\alpha}) - \beta\alpha^{1/2}Y_5(\beta\sqrt{\alpha})],\\ \varepsilon_3 &= [8I_4(\beta\sqrt{\alpha}) + \beta\alpha^{1/2}I_5(\beta\sqrt{\alpha})], \quad \varepsilon_4 = [8K_4(\beta\sqrt{\alpha}) - \beta\alpha^{1/2}K_5(\beta\sqrt{\alpha})],\end{aligned} \tag{34b}$$

$$\begin{aligned}\{C_R\} &= \{C_{n+1,1} \quad C_{n+1,2} \quad C_{n+1,3} \quad C_{n+1,4}\}\\ &= \{\bar{C}_{4n+1} \quad \bar{C}_{4n+2} \quad \bar{C}_{4n+3} \quad \bar{C}_{4n+4}\},\end{aligned} \tag{35}$$

$$\bar{C}_{4n+1} = C_{n+1,1},\ \bar{C}_{4n+2} = C_{n+1,2},\ \bar{C}_{4n+3} = C_{n+1,3},\ \bar{C}_{4n+4} = C_{n+1,4}, \tag{36}$$

$$p = 5n + 4. \tag{37}$$

In the last equations, p represents the total number of equations. From the above derivations one sees that, from each attaching point for a spring–mass system, one may obtain five equations (including three compatibility equations, one force-equilibrium equation and one governing equation for the sprung mass), and from each boundary (Ⓛ or Ⓡ) one may obtain two equations. Hence, for a beam carrying n spring–mass systems, the total number of equations that one may obtain for the integration constants C_{vi} and mode displacements Z_v ($v = 1$–n, $i = 1$–4) is equal to $5n + 4$, i.e., $p = 5n + 4$ as shown by equation (37). Of course, the total number of unknowns (C_{vi} and Z_v) is also equal to $5n + 4$. From equation (8) one sees that the solution $\bar{Y}_v(\xi)$for each beam segment contains four unknown integration constants C_{vi} ($i = 1$–4) and from equation (17) one sees that the governing equation for each sprung mass contains one additional unknown Z_v, hence if a beam is carrying n sprung masses, then the total number of the beam segments is $n + 1$ and thus the total number of unknowns (C_{vi} and Z_v) is equal to $4(n + 1) + n = 5n + 4 = p$.

6. OVERALL COEFFICIENT MATRIX $[\bar{B}]$ OF THE ENTIRE BEAM AND THE FREQUENCY EQUATION

If all the unknowns C_{vi} and Z_v ($v = 1$–n, $i = 1$–4) are replaced by a column vector $\{\bar{C}\}$ with coefficients $\bar{C}_k$ ($k = 1, 2, \ldots, p$) defined by equations (30), (24b) and (36), then the matrices $[B_L]$, $[B_v]$ and $[B_R]$ are similar to the element property matrices (for the finite element method) with corresponding identification numbers of degrees of freedom (d.o.f.) shown on the top side and right side of the matrices defined by equations (28), (24c) and (34). Basing on the assembly technique for the direct stiffness matrix method, it is easy to arrive at the following coefficient equation for the entire vibrating system:

$$[\bar{B}]\{\bar{C}\} = 0. \tag{38}$$

Non-trivial solution of the last equation requires that

$$|\bar{B}| = 0 \tag{39}$$

which is the frequency equation, and the half-interval technique [23] may be used to solve the eigenvalues $\bar{\omega}_j$ ($j = 1, 2, \ldots$). To substitute each value of $\bar{\omega}_j$ into equation (38), the values of unknowns $\bar{C}_k$ ($k = 1, 2, \ldots, p$) can be determined. From equation (30), one sees that $\bar{C}_{4v-3} = C_{v1}$, $\bar{C}_{4v-2} = C_{v2}$, $\bar{C}_{4v-1} = C_{v3}$, $\bar{C}_{4v} = C_{v4}$, $v = 1$–n, hence the substitution of C_{vi} ($i = 1$–4) into equation (8) will define the corresponding mode shape $\bar{Y}^{(j)}(\xi)$. For a cantilever beam carrying one ($n = 1$) and two ($n = 2$) spring–mass systems, the corresponding overall coefficient matrices $[\bar{B}]_{(1)}$ and $[\bar{B}]_{(2)}$ were shown in Appendix A [see equations (A1) and (A2)]. From the lengthy expressions one sees that the existing explicit formulations are not suitable for a beam carrying more than two ($n > 2$) spring–mass systems. However, this is not true for the numerical assembly method (NAM) presented in this paper.

7. COEFFICIENT MATRICES $[B_L]$ AND $[B_R]$ FOR VARIOUS BOUNDARY CONDITIONS

From the previous sections, it is seen that the form of the coefficient matrix $[B_v]$ for each attaching point of the spring–mass system has nothing to do with the boundary conditions of the beam. Hence for a "constrained" beam with various supporting conditions, the only thing one should do is to modify the values of the two boundary matrices $[B_L]$ and $[B_R]$ defined by equations (28) and (34), respectively, according to the actual boundary conditions. Thus, the same numerical assembly procedures presented in the last section may be followed. This is one of the predominant advantages of the NAM. The boundary matrices $[B_L]$ and $[B_R]$ for various boundary conditions are given in Appendix B at the end of this paper.

8. NUMERICAL RESULTS AND DISCUSSION

The dimensions and physical properties of the non-uniform beam studied in this paper are: beam length $L = 40$ in, Young's modulus $E = 3{\cdot}0 \times 10^7$ psi, cross-sectional area at the shallow end of the beam $A_0 = 1{\cdot}5\,\text{in}^2$, area moment of inertia at the shallow end of the beam $I_0 = 0{\cdot}28125\,\text{in}^4$, mass density of beam material $\rho = 0{\cdot}283\,\text{lbm}$, taper ratio of the beam $\alpha = 2{\cdot}0$, total mass of the beam $m_b = \rho A_0 L\,[\frac{1}{3}(\alpha - 1)^2 + (\alpha - 1) + 1] = 29{\cdot}715\,\text{lbm}$ and reference stiffness for the beam $k_b = EI_0/L^3 = 312{\cdot}5\,\text{lbf/in}$. For convenience, two non-dimensional parameters for each spring–mass system were introduced: $m_i^* = m_i/m_b$ and $k_i^* = k_i/k_b$, $i = 1, 2, \ldots$, besides, the two-letter acronyms FC, CF, SC, CS, CC and SS were used to denote the free–clamped, clamped–free, simply supported–clamped, clamped–simply supported, clamped–clamped, and simply supported–simply supported boundary conditions of the beam respectively.

8.1. A NON-UNIFORM BEAM CARRYING ONE SPRING–MASS SYSTEM

For convenience of comparison, the free vibration analysis on the bare (or unconstrained) non-uniform beam which was obtained from the present approach is very close to those obtained from FEM. The FEM model is shown in Figure 2, where the entire non-uniform beam is replaced by a stepped beam composed of 40 uniform beam segments. The cross-sectional area A_i and the moment of inertia I_i of the ith uniform beam segment are determined from the average cross-sectional dimensions of the corresponding ith non-uniform beam segment, respectively, and the mass per unit length of the ith uniform beam segment is evaluated by ρA_i. The length of each uniform beam segment is $l = L/40 = 1{\cdot}0$ in. Table 1 shows the lowest five natural frequencies with the six types of boundary conditions (i.e., FC, CF, SC, CS, CC and SS) and Figure 3(a)–3(f) show the corresponding mode shapes. From Figure 3 one sees that, for the SC, CS, CC, and SS boundary conditions, the modal displacements near the left end of the beam are larger than those near the right end of the beam. This is a reasonable result, because the stiffness of the left end is much smaller than that of the right end for the non-uniform beam shown in Figure 1.

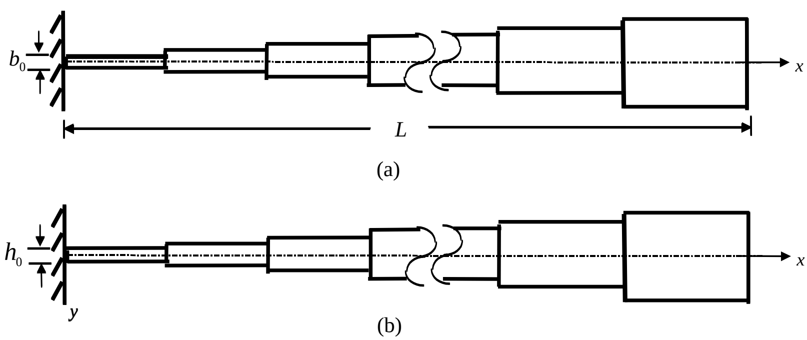

Figure 2. The finite element model for non-uniform beam: (a) top view and (b) front view.

TABLE 1

The lowest five natural frequencies ω_j ($j = 1$–5) for the bare non-uniform beam (without carrying any spring-mass system)

Boundary conditions	Methods	Natural frequencies (rad/s)				
		ω_1	ω_2	ω_3	ω_4	ω_5
FC	NAM†	25·77532	108·93610	270·72329	511·65966	832·52916
	FEM†	25·76465	108·87999	270·57492	511·37488	832·06822
CF	NAM	7·23408	73·47681	236·81331	477·59822	798·39106
	FEM	7·23856	73·51321	236·92092	477·81289	798·75291
SC	NAM	71·61388	212·87668	433·85401	734·70179	1115·57882
	FEM	71·59223	212·80942	433·71599	734·47104	1115·24149
CS	NAM	53·77585	196·23185	417·05549	717·82045	1098·63870
	FEM	53·79692	196·30255	417·20453	718·07951	1099·04656
CC	NAM	91·83540	251·75856	492·37771	813·02661	1213·79486
	FEM	91·85148	251·80279	492·46522	813·17648	1214·03442
SS	NAM	38·76810	162·22786	363·50517	644·48275	1005·41779
	FEM	38·76239	162·20436	363·45241	644·39225	1005·28683

†NAM = numerical assembly method; FEM = finite element method.

If $\Delta\omega = \omega_j - \omega_{j-1}$ ($j = 1$–5) denotes the difference between any two adjacent natural frequencies and N_j denotes the node number of jth mode shape, then from Table 1 it is seen that the minimum values of $\Delta\omega$ for the FC, CF, SC, CS, CC and SS beams are approximately equal to 83, 66, 141, 142, 160 and 123 rad/s, respectively. Furthermore, from Figure 3(a)–3(f), it is found that $N_j = j - 1$, i.e., the "node' number N_j is always less than the "mode" number j by 1, for the bare non-uniform beam with the six boundary conditions. However, this is not true for the constrained non-uniform beam as shown in Figure 4(a)–4(f): either the values of $\bar{\omega}$ or those of $\Delta\bar{\omega}$ for the constrained non-uniform beam decrease significantly as may be seen from Table 2.

The percentage differences between $\bar{\omega}_{jNAM}$ and $\bar{\omega}_{jFEM}$ shown in the parentheses ( ) of Table 2 were calculated using the formula: $\Delta_j = (\bar{\omega}_{jNAM} - \bar{\omega}_{jFEM}) \times 100\%/\bar{\omega}_{jNAM}$, where $\bar{\omega}_{jNAM}$ and $\bar{\omega}_{jFEM}$ denote the jth natural frequencies of the constrained non-uniform beam obtained from the presented NAM and the conventional FEM, respectively. From Table 2 one finds that the maximum value of Δ_j is $\Delta_5 = 0{\cdot}0006\%$ (for the FC boundary condition), hence the accuracy of the NAM is excellent.

From the mode shapes of the beam with six types of boundary conditions as shown in Figure 4(a)–4(f), one sees that the first mode shapes $\bar{Y}_1(\xi)$ are very close to the second ones $\bar{Y}_2(\xi)$, but this does not mean that the corresponding 1st natural frequencies ($\bar{\omega}_1$) are approximately equal to the 2nd ones ($\bar{\omega}_2$) as shown in Table 2, besides, the relationship between the node number N_j and the mode number j is given as $N_j = j - 2$ ($j \geqslant 2$).

8.2. A NON-UNIFORM BEAM CARRYING THREE SPRING-MASS SYSTEMS

If all the situations are exactly the same as the last example and the only difference is that the one spring–mass system was replaced by the three spring–mass systems with locations (x/L), magnitudes of point masses ($m_i^* = m_i/m_b$) and spring constants ($k_i^* = k_i/k_b$) as shown

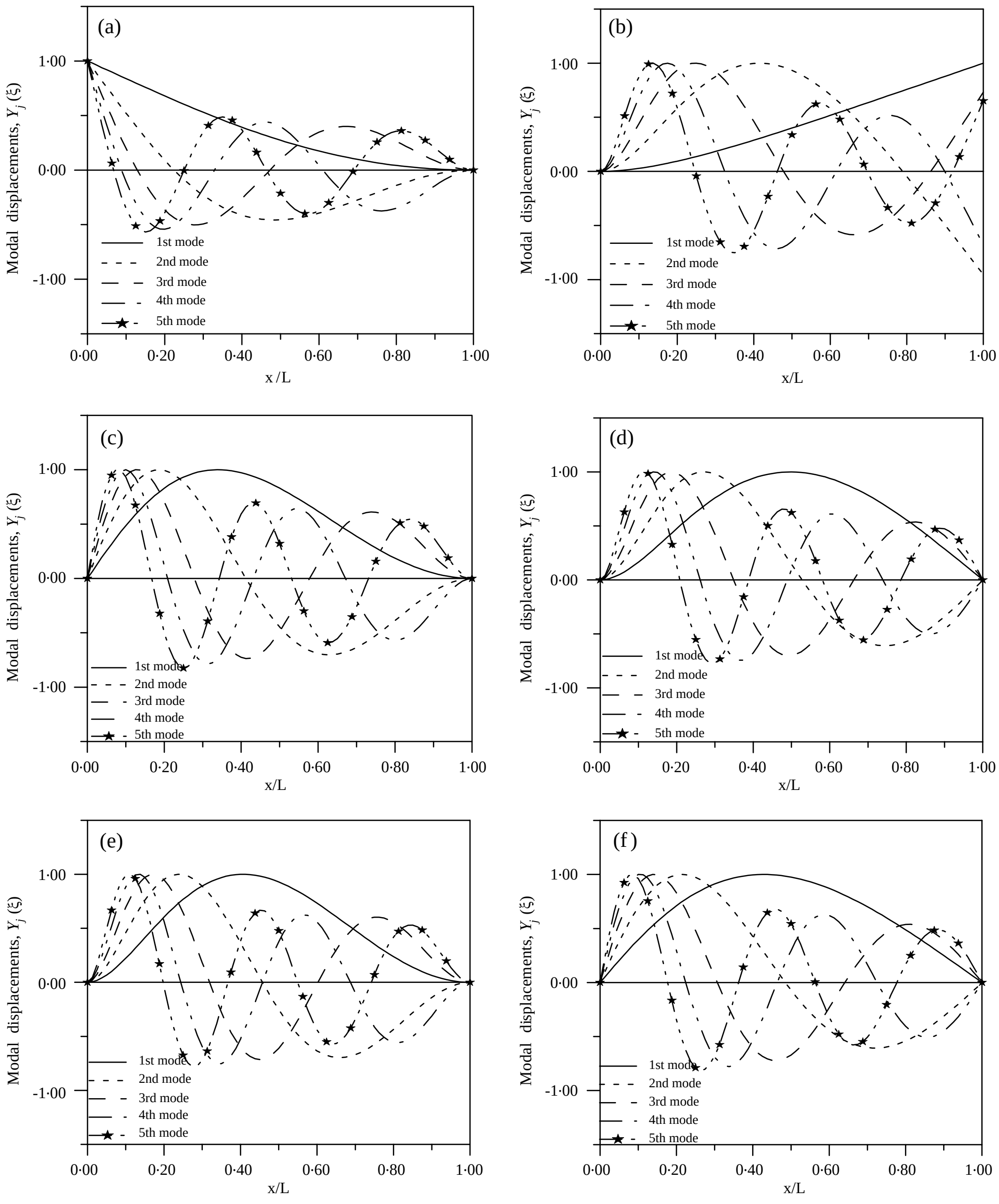

Figure 3. The lowest five mode shapes $Y_j(\xi)$ (j = 1–5) for the bare non-uniform beam (without carrying any spring–mass system) with the support conditions: (a) FC, (b) CF, (c) SC, (d) CS, (e) CC and (f) SS.

in Table 3, then the lowest five natural frequencies $\bar{\omega}_j$ (j = 1–5) and the corresponding mode shapes $\bar{Y}_j(\xi)$ (j = 1–5) for the six boundary conditions are shown in Table 4 and Figure 5(a)–5(f), respectively.

A comparison between Tables 2 and 4 reveals that the natural frequencies of the tapered beam carrying three spring–mass systems are much lower than the corresponding ones of the tapered beam carrying one spring–mass system. For this reason, the corresponding mode shapes for the tapered beam carrying three spring–mass systems [see Figure 5(a)–5(f)]

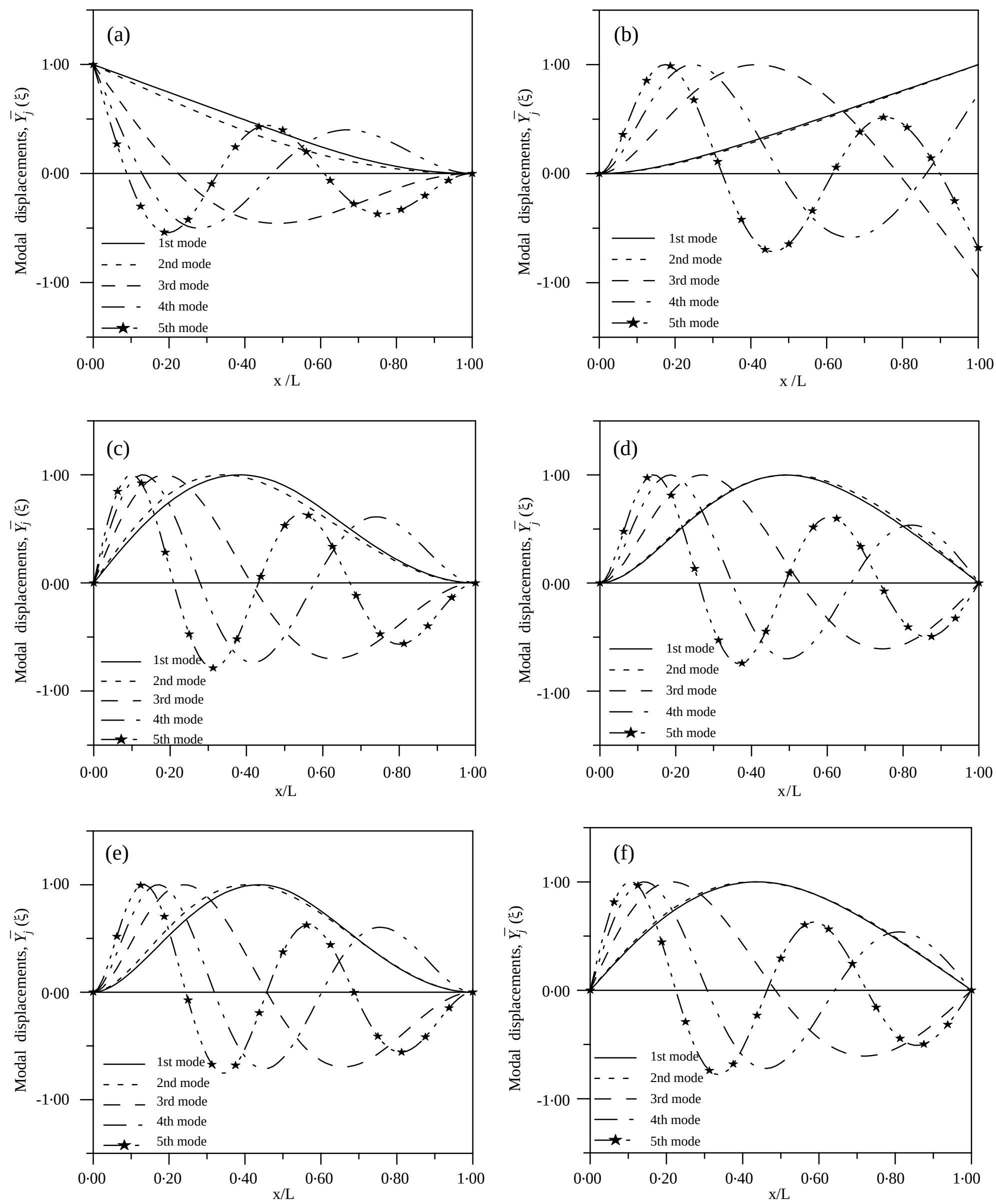

Figure 4. The lowest five mode shapes $\bar{Y}_j(\xi)$ (j = 1–5) for the non-uniform beam carrying one spring–mass system with non-dimensional point mass $m_1^* = m_1/m_b = 0{\cdot}2$ and non-dimensional spring constant $k_1^* = k_1/k_b = 3{\cdot}0$ located at $x_1/L = 0{\cdot}5$ with the support conditions: (a) FC, (b) CF, (c) SC, (d) CS, (e) CC and (f) SS.

are very different from those for the tapered beam carrying one spring–mass system [see Figure 4(a)–4(f)].

From Table 4, one finds that the maximum value of percentage differences between $\bar{\omega}_{jNAM}$ and $\bar{\omega}_{jFEM}$ (j = 1–5) is $\Delta_5 = 0{\cdot}0005\%$ (for the FC and CF boundary conditions), i.e., the accuracy of the NAM is not affected by the total number of the attached spring–mass systems.

TABLE 2

The lowest five natural frequencies $\bar{\omega}_j (j = 1\text{–}5)$ for the non-uniform beam carrying one spring–mass system (located at $x_1/L = 0{\cdot}5$ with $k_1^* = k_1/k_b = 3{\cdot}0$ and $m_1^* = m_1/m_b = 0{\cdot}2$)

Boundary conditions	Methods	Natural frequencies (rad/s) $\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
FC	NAM	7·02596	25·89419	109·03110	270·72514	511·67660
		(0·000004%)	(0·0004%)	(0·0005%)	(0·0005%)	(0·0006%)
	FEM	7·02593	25·88356	108·97503	270·57671	511·39183
CF	NAM	6·19333	8·23570	73·61888	236·81448	477·61686
		(− 0·0003%)	(− 0·0003%)	(− 0·0005%)	(− 0·0005%)	(− 0·0005%)
	FEM	6·19545	8·23798	73·65525	236·92211	477·83184
SC	NAM	7·04981	71·75787	212·89407	433·86578	734·71042
		(0·000001%)	(0·0003%)	(0·0003%)	(0·0003%)	(0·0003%)
	FEM	7·04980	71·73625	212·82678	433·72766	734·47976
CS	NAM	7·03733	53·98203	196·23381	417·08009	717·82045
		(− 0·000002%)	(− 0·0004%)	(− 0·0004%)	(− 0·0004%)	(− 0·0004%)
	FEM	7·03735	54·00303	196·30450	417·22917	718·07978
CC	NAM	7·08395	91·97187	251·76321	492·39393	813·03060
		(0·0000%)	(− 0·0002%)	(− 0·0002%)	(− 0·0002%)	(− 0·0002%)
	FEM	7·05395	91·98794	251·80735	492·48155	813·18046
SS	NAM	7·01473	39·04175	162·22841	363·53090	644·48598
		(0·000003%)	(0·0001%)	(0·0001%)	(0·0001%)	(0·0001%)
	FEM	7·01471	39·03607	162·20488	363·47824	644·39528

Note: The percentage differences between $\bar{\omega}_{jNAM}$ and $\bar{\omega}_{jFEM}$ shown in the parentheses ( ) were determined using the formula: $\Delta_j = (\bar{\omega}_{jNAM} - \bar{\omega}_{jFEM}) \times 100\%/\bar{\omega}_{jNAM}$.

TABLE 3

The locations and magnitudes of the three spring–mass systems on the non-uniform beam

Locations x_i/L			Magnitudes of spring constants $k_i^* = k_i/k_b$			Magnitudes of point masses $m_i^* = m_i/m_b$		
x_1/L	x_2/L	x_3/L	k_1^*	k_2^*	k_3^*	m_1^*	m_2^*	m_3^*
0·1	0·5	0·9	3·0	4·5	6·0	0·2	0·5	1·0

8.3. A NON-UNIFORM BEAM CARRYING FIVE SPRING–MASS SYSTEMS

The present example is also the same as the last one except that the three spring–mass systems were replaced by five ones. The locations (x_i/L) and the magnitudes of the five attachments ($m_i^* = m_i/m_b$ and $k_i^* = k_i/k_b$, $i = 1\text{–}5$) are shown in Table 5. The lowest five natural frequencies $\bar{\omega}_j$ ($j = 1\text{–}5$) and the corresponding mode shapes $\bar{Y}_j(\xi)$ ($j = 1\text{–}5$) are shown in Table 6 and Figure 6(a)–6(f), respectively, for the six boundary conditions (FC, CF, SC, CS, CC and SS).

Table 6 shows that $\bar{\omega}_1 \cong 4{\cdot}4$, $\bar{\omega}_2 \cong 5{\cdot}0$, $\bar{\omega}_3 \cong 5{\cdot}4$, $\bar{\omega}_4 \cong 6{\cdot}1$ and $\bar{\omega}_5 \cong 6{\cdot}8$ rad/s for all the six boundary conditions except that $\bar{\omega}_1 \cong 2{\cdot}9$ rad/s for the CF beam. In other words, for

Table 4

The lowest five natural frequencies $\bar{\omega}_j$ ($j = 1$–5) for the non-uniform beam carrying three spring–mass systems with parameters as shown in Table 3

Boundary conditions	Methods	Natural frequencies (rad/s)				
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
FC	NAM	4·46793	5·42430	6·77941	27·02954	109·20376
		(0·0000%)	(0·000006%)	(0·00002%)	(0·0004%)	(0·0005%)
	FEM	4·46793	5·42427	6·77929	27·01896	109·14758
CF	NAM	3·14514	5·34475	7·06218	10·48115	73·77557
		(− 0·0003%)	(− 0·00002%)	(− 0·000003%)	(− 0·0003%)	(− 0·0005%)
	FEM	3·14604	5·34485	7·06220	10·48446	73·81192
SC	NAM	4·46797	5·45499	7·05721	71·87890	212·95655
		(0·0000%)	(0·000002%)	(0·0000%)	(0·0003%)	(0·0003%)
	FEM	4·46797	5·45498	7·05721	71·85733	212·88931
CS	NAM	4·46453	5·44082	7·06285	54·12206	196·26949
		(0·0000%)	(− 0·000006%)	(− 0·000001%)	(− 0·0004%)	(− 0·0004%)
	FEM	4·46453	5·44085	7·06286	54·14301	196·34007
CC	NAM	4·46798	5·45977	7·06313	92·04866	251·78398
		(0·0000%)	(0·0000%)	(− 0·000001%)	(− 0·0002%)	(− 0·0002%)
	FEM	4·46798	5·45977	7·06314	92·06470	251·82793
SS	NAM	4·46300	5·41537	7·05322	39·25189	162·30375
		(0·000002%)	(0·000004%)	(0·0000%)	(0·0003%)	(0·0001%)
	FEM	4·46299	5·41535	7·05322	39·24623	162·28025

Note: The percentage differences between $\bar{\omega}_{jNAM}$ and $\bar{\omega}_{jFEM}$ shown in the parentheses ( ) were determined using the formula: $\Delta_j = (\bar{\omega}_{jNAM} - \bar{\omega}_{jFEM}) \times 100\%/\bar{\omega}_{jNAM}$.

a tapered beam carrying five spring–mass systems with locations and magnitudes as shown in Table 5, the lowest five natural frequencies $\bar{\omega}_j (j = 1$–$5)$ are almost independent of the boundary conditions of the beam. This may be the reason why the lowest five mode shapes for the SS beam resemble like those for the CC, SC or CS beam. It is noted that all the lowest five mode shapes do not have any node as shown in Figure 6(a)–6(f). From Table 6, we can see that the maximum value of percentage differences between $\bar{\omega}_{jNAM}$ and $\bar{\omega}_{jFEM}$ ($j = 1$–5) is $\Delta_1 = -0{\cdot}0003\%$ (CF beam), hence the accuracy of the NAM is also excellent for the present case.

9. CONCLUSIONS

(1) For a double-tapered beam with various boundary conditions and carrying any number of spring–mass systems, the exact solutions for the natural frequencies and the corresponding mode shapes are easily obtained by using with the numerical assembly method (NAM). It has been found that the locations and magnitudes of the attached spring–mass systems significantly affect the free vibration responses of the beam.

(2) For an unconstrained non-uniform beam, if the stiffness of the left part is much smaller than that of the right part, then, for the SC, CS, CC, and SS boundary

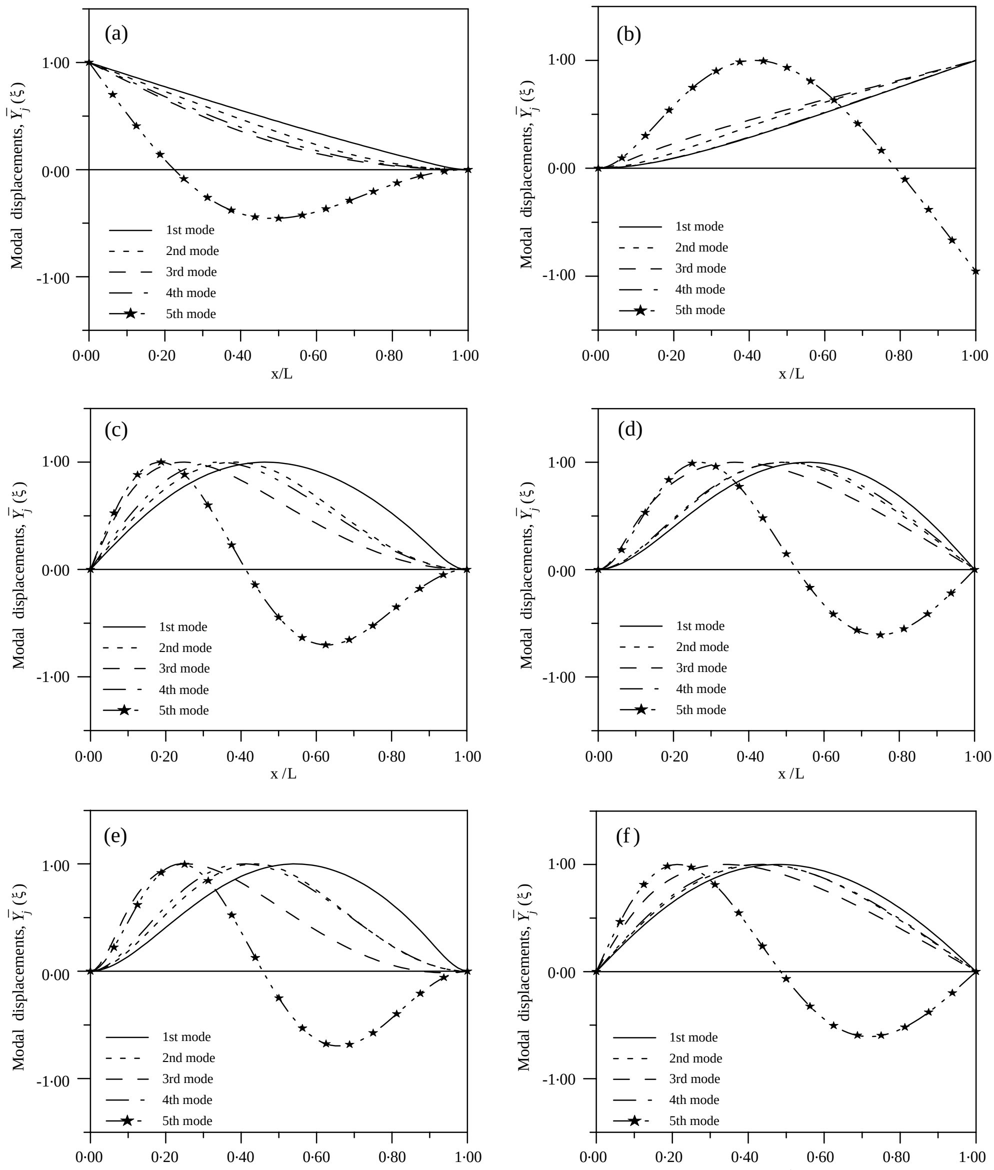

Figure 5. The lowest five mode shapes $\bar{Y}_j(\xi)$ $(j = 1\text{–}5)$ for the non-uniform beam carrying three spring–mass systems with locations and magnitudes as shown in Table 3 with the support conditions: (a) FC, (b) CF, (c) SC, (d) CS, (e) CC and (f) SS.

conditions, the modal displacements near the left end are larger than the corresponding ones near the right end of the beam.

(3) For the non-uniform beam with six boundary conditions studied in this paper, the attachment of the spring–mass system(s) reduces the lowest five natural frequencies of the beam significantly. For this reason, the configuration of the corresponding mode shapes also varies predominantly compared with those of the associated bare beam.

TABLE 5

The locations and magnitudes of the five spring–mass systems on the non-uniform beam

Locations x_i/L					Magnitudes of spring constants $k_i^* = k_i/k_b$					Magnitudes of point masses $m_i^* = m_i/m_b$				
x_1/L	x_2/L	x_3/L	x_4/L	x_5/L	k_1^*	k_2^*	k_3^*	k_4^*	k_5^*	m_1^*	m_2^*	m_3^*	m_4^*	m_5^*
0.1	0.3	0.5	0.7	0.9	3.0	3.5	4.5	5.0	6.0	0.2	0.3	0.5	0.65	1.0

TABLE 6

The lowest five natural frequencies $\bar{\omega}_j$ ($j = 1$–5) for the non-uniform beam carrying five spring–mass systems with parameters as shown in Table 5

Boundary conditions	Methods	Natural frequencies (rad/s)				
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
FC	NAM	4·46792	5·04896	5·41877	6·08576	6·82670
		(0·00000%)	(0·000002%)	(0·000007%)	(0·00001%)	(0·00001%)
	FEM	4·46792	5·04895	5·41873	6·08569	6·82663
CF	NAM	2·96451	4·87707	5·39880	6·19975	7·06252
		(− 0·0003%)	(− 0·00001%)	(− 0·00001%)	(− 0·00001%)	(− 0·00001%)
	FEM	2·96548	4·87713	5·39884	6·19980	7·06253
SC	NAM	4·46796	5·05364	5·45487	6·20894	7·05738
		(0·0000%)	(0·0000%)	(0·000004%)	(0·000002%)	(0·0000%)
	FEM	4·46796	5·05364	5·45485	6·20893	7·05738
CS	NAM	4·46442	5·03691	5·44194	6·21220	7·06288
		(0·0000%)	(− 0·000002%)	(− 0·000004%)	(− 0·000005%)	(− 0·000001%)
	FEM	4·46442	5·03692	5·44196	6·21223	7·06289
CC	NAM	4·46798	5·05449	5·45977	6·21903	7·06314
		(0·0000%)	(0·0000%)	(0·0000%)	(− 0·000002%)	(− 0·000001%)
	FEM	4·46798	5·05449	5·45978	6·21904	7·06315
SS	NAM	4·46270	5·02262	5·41794	6·18725	7·05394
		(0·0000%)	(0·000004%)	(0·000004%)	(0·000002%)	(0·000001%)
	FEM	4·46270	5·02260	5·41792	6·18724	7·05393

Note: The percentage differences between $\bar{\omega}_{jNAM}$ and $\bar{\omega}_{jFEM}$shown in the parentheses ( ) were determined using the formula: $\Delta_j = (\bar{\omega}_{jNAM} - \bar{\omega}_{jFEM}) \times 100\%/\bar{\omega}_{jNAM}$.

(4) In addition to the spring–mass systems studied in this paper, the presented NAM is also available for a beam (either uniform or non-uniform) carrying any number of the other concentrated attachments, such as rigidly attached point masses, linear springs and/or rotational springs. One of the main advantages of the NAM being superior to the FEM is that the NAM solutions are exact and the FEM solutions are approximate. It is believed that the availability of the NAM to the plate or shell problems should be definite and worthy of further studies.

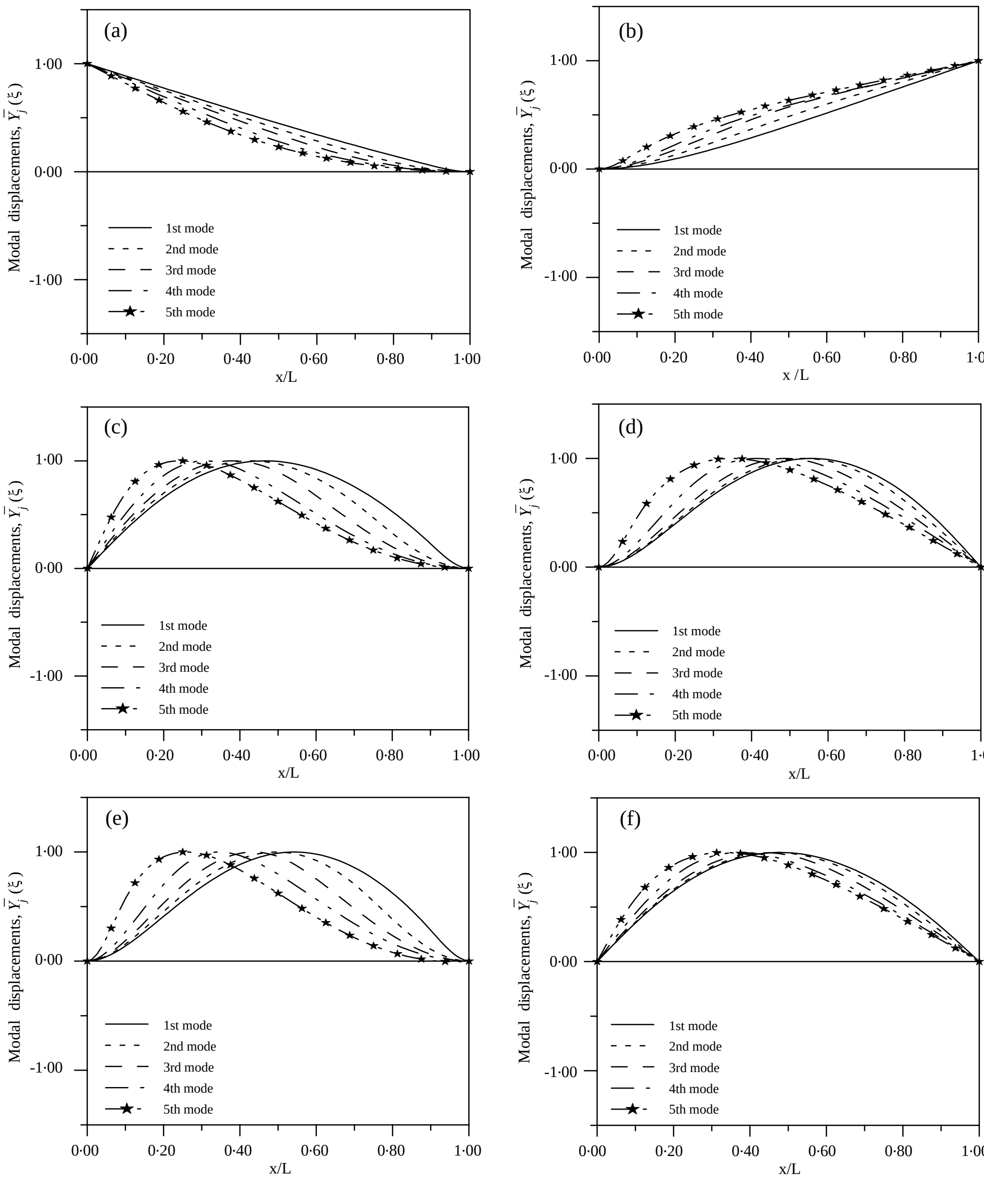

Figure 6. The lowest five mode shapes $\bar{Y}_j(\xi)$ ($j = 1$–5) for the non-uniform beam carrying five spring–mass systems with locations and magnitudes as shown in Table 5 with the support conditions: (a) FC, (b) CF, (c) SC, (d) CS, (e) CC and (f) SS.

REFERENCES

1. P. A. A. LAURA, M. J. MAURIZI and J. L. POMBO 1975 *Journal of Sound and Vibration* **41**, 397–405. A note on the dynamic analysis of an elastically restrained-free beam with a mass at the free end.
2. P. A. A. LAURA, M. J. MAURIZI and J. L. POMBO, L. E. LUISONI and R. GELOS 1977 *Applied Acoustic* **10**, 121–145. On the dynamic behavior of structural elements carrying elastically mounted concentrated masses.

3. M. GURGOZE 1984 *Journal of Sound and Vibration* **96**, 461–468. A note on the vibrations of restrained beam and rods with point masses.
4. P. A. A. LAURA, C. P. FILIPICH and V. H. CORTINEZ 1987 *Journal of Sound and Vibration* **112**, 177–182. Vibration of beams and plates carrying concentrated masses.
5. J. S. WU and T. L. LIN 1990 *Journal of Sound and Vibration* **136**, 201–213. Free vibration analysis of a uniform cantilever beam with point masses by an analytical-and-numerical-combined method.
6. M. N. HAMDAN and B. A. JUBRAN 1991 *Journal of Sound and Vibration* **150**, 203–216. Free and forced vibrations of a restrained uniform beam carrying an intermediate lumped mass and a rotary inertia.
7. R. E. ROSSI, P. A. A. LAURA, D. R. AVALOS and H. LARRONDO 1993 *Journal of Sound and Vibration* **165**, 209–223. Free vibration of Timoshenko beams carrying elastically mounted.
8. M. GURGOZE 1998 *Journal of Sound and Vibration* **217**, 585–595. On the alternative formulations of the frequency equations of a Bernoulli–Euler beam to which several spring–mass systems are attached inspan.
9. J. S. WU and H. M. CHOU 1998 *Journal of Sound and Vibration* **213**, 317–332. Free vibration analysis of a cantilever beam carrying any number of elastically mounted point masses with the analytical-and-numerical-combined method.
10. J. S. WU and H. M. CHOU 1999 *Journal of Sound and Vibration* **220**, 451–468. A new approach for determining the natural frequencies and mode shapes of a uniform beam carrying any number of sprung masses.
11. J. S. WU, D. W. CHEN and H. M. CHOU 1999 *International Journal for Numerical Methods in Engineering* **45**, 1277–1295. On the eigenvalues of a uniform cantilever beam carrying any number of spring–damper-mass system.
12. J. S. WU and D. W. CHEN 2001 *International Journal for Numerical Methods in Engineering* **50**, 1039–1058. Free vibration analysis of a Timoshenko beam carrying multiple spring–mass systems by using the numerical assembly technique.
13. G. W. HOUSNER and W. O. KEIGHTLEY 1962 *Journal of Engineering Mecahnics Division, Proceedings of the American Society of Civil Engineers* **88**, EM2, 95-123. Vibrations of linearly tapered cantilever beams.
14. A. C. HEIDEBRECHT 1967 *Journal of Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers* **93**, EM2, 1–15. Vibrations of non-uniform simply-supported beams.
15. A. K. GUPTA 1985 *Journal of Structural Engineering* **111**, 19–15. Vibration of tapered beams.
16. S. ABRATE 1995 *Journal of Sound and Vibration* **185**, 703–716. Vibration of non-uniform rods and beams.
17. S. NAGULESWARAN 1996 *Journal of Sound and Vibration* **195**, 331–337. Comments on vibration of non-uniform rods and beams.
18. M. A. DE ROSA and N. M. AUCIELLO 1996 *Computers and structures* **60**, 197–202. Free vibrations of tapered beams with flexible ends.
19. J. S. WU and M. HSIEH 2000 *Structural Engineering and Mechanics* **9**, 449–467. Free vibration analysis of a non-uniform beam with multiple point masses.
20. DANIEL I. GORMAN 1975 *Free Vibration Analysis of Beams and Shafts*. New York: John Wiley & Sons, Inc.
21. T. V. KARMAN and M. A. BIOT 1940 *Mathematical Methods in Engineering* New York: McGraw-Hill.
22. P. A. A. LAURA, E. A. SUSEMIHL, L. E. POMBO, L. E. LUISONI and R. GELOS 1977 *Applied Acoustics* **10**, 121–145. On the dynamic behavior of structural elements carrying elastically mounted, concentrated masses.
23. J. D. FAIRES and R. L. BURDEN 1993 *Numerical Methods*. Boston, USA: PWS Publishing Company.

APPENDIX A

The overall coefficient matrix for a non-uniform cantilever beam carrying one spring–mass system, $[B]_{(1)}$, and that carrying two spring–mass systems, $[B]_{(2)}$, are given in their explicit forms by equations (A1) and (A2) respectively.

$$
[B]_{(1)} = \begin{array}{c} \\ \end{array}
\begin{array}{ccccccccc|c}
\bar{C}_1 & \bar{C}_2 & \bar{C}_3 & \bar{C}_4 & \bar{C}_5 & \bar{C}_6 & \bar{C}_7 & \bar{C}_8 & \bar{C}_9 & \\
J_2(\beta) & Y_2(\beta) & I_2(\beta) & K_2(\beta) & 0 & 0 & 0 & 0 & 0 & 1 \\
J_3(\beta) & Y_3(\beta) & -I_3(\beta) & K_3(\beta) & 0 & 0 & 0 & 0 & 0 & 2 \\
J_2(\delta_1) & Y_2(\delta_1) & I_2(\delta_1) & K_2(\delta_1) & -J_2(\delta_1) & -Y_2(\delta_1) & -I_2(\delta_1) & -K_2(\delta_1) & 0 & 3 \\
J_3(\delta_1) & Y_3(\delta_1) & -I_3(\delta_1) & K_3(\delta_1) & -J_3(\delta_1) & -Y_3(\delta_1) & I_3(\delta_1) & -K_3(\delta_1) & 0 & 4 \\
J_4(\delta_1) & Y_4(\delta_1) & I_4(\delta_1) & K_4(\delta_1) & -J_4(\delta_1) & -Y_4(\delta_1) & -I_4(\delta_1) & -K_4(\delta_1) & 0 & 5 \\
\Delta_{11} & \Delta_{21} & \Delta_{31} & \Delta_{41} & -\Delta_{51} & -\Delta_{61} & -\Delta_{71} & -\Delta_{81} & 0 & 6 \\
\xi_1^{-1}J_2(\delta_1) & \xi_1^{-1}Y_2(\delta_1) & \xi_1^{-1}I_2(\delta_1) & \xi_1^{-1}K_2(\delta_1) & 0 & 0 & 0 & 0 & -1+\gamma_1^2 & 7 \\
0 & 0 & 0 & 0 & J_4(\beta\sqrt{\alpha}) & Y_4(\beta\sqrt{\alpha}) & I_4(\beta\sqrt{\alpha}) & K_4(\beta\sqrt{\alpha}) & 0 & 8 \\
0 & 0 & 0 & 0 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & 0 & 9
\end{array}
\tag{A1}
$$

where

$$\varepsilon_1 = [8J_4(\beta\sqrt{\alpha}) - \beta\alpha^{1/2}J_5(\beta\sqrt{\alpha})], \quad \varepsilon_2 = [8Y_4(\beta\sqrt{\alpha}) - \beta\alpha^{1/2}Y_5(\beta\sqrt{\alpha})],$$

$$\varepsilon_3 = [8I_4(\beta\sqrt{\alpha}) + \beta\alpha^{1/2}I_5(\beta\sqrt{\alpha})], \quad \varepsilon_4 = [8K_4(\beta\sqrt{\alpha}) - \beta\alpha^{1/2}K_5(\beta\sqrt{\alpha})]$$

$$\beta = 2L\Omega/(\alpha-1), \quad \Delta_{11} = 8\beta^2 J_4(\beta\sqrt{\xi_1}) - \beta^3\xi_1^{1/2}J_5(\beta\sqrt{\xi_1}) + 8\theta_1\xi_1^{-2}J_2(\beta\sqrt{\xi_1}),$$

$$\delta_1 = \beta\sqrt{\xi_1}, \quad \Delta_{21} = 8\beta^2 Y_4(\beta\sqrt{\xi_1}) - \beta^3\xi_1^{1/2}Y_5(\beta\sqrt{\xi_1}) + 8\theta_1\xi_v^{-2}Y_2(\beta\sqrt{\xi_1}),$$

$$\Delta_{31} = 8\beta^2 I_4(\beta\sqrt{\xi_1}) + \beta^3\xi_1^{1/2}I_5(\beta\sqrt{\xi_1}) + 8\theta_1\xi_1^{-2}I_2(\beta\sqrt{\xi_1}),$$

$$\Delta_{41} = 8\beta^2 K_4(\beta\sqrt{\xi_1}) - \beta^3\xi_1^{1/2}K_5(\beta\sqrt{\xi_1}) + 8\theta_1\xi_1^{-2}K_2(\beta\sqrt{\xi_1}),$$

$$\Delta_{51} = 8\beta^2 J_4(\beta\sqrt{\xi_1}) - \beta^3\xi_1^{1/2}J_5(\beta\sqrt{\xi_1}), \quad \Delta_{61} = 8\beta^2 Y_4(\beta\sqrt{\xi_1}) - \beta^3\xi_1^{1/2}Y_5(\beta\sqrt{\xi_1})$$

$$\Delta_{71} = 8\beta^2 I_4(\beta\sqrt{\xi_1}) + \beta^3\xi_1^{1/2}I_5(\beta\sqrt{\xi_1}), \quad \Delta_{81} = 8\beta^2 K_4(\beta\sqrt{\xi_1}) - \beta^3\xi_1^{1/2}K_5(\beta\sqrt{\xi_1})$$

$$\theta_1 = \left\{ \frac{m_1^*[\frac{1}{3}(\alpha-1)^2 + (\alpha-1) + 1](\Omega L)^4}{1 - m_1^*[\frac{1}{3}(\alpha-1)^2 + (\alpha-1) + 1](\Omega L)^4/k_1^*} \frac{1}{(\alpha-1)^3} \right\},$$

$$\gamma_1^2 = \frac{m_1^*[\frac{1}{3}(\alpha-1)^2 + (\alpha-1) + 1]}{k_1^*} \cdot (\Omega L)^4$$

$$
[B]_{(2)} = \left[\begin{array}{cccccccccccccc}
\bar{C}_1 & \bar{C}_2 & \bar{C}_3 & \bar{C}_4 & \bar{C}_5 & \bar{C}_6 & \bar{C}_7 & \bar{C}_8 & \bar{C}_9 & \bar{C}_{10} & \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & \bar{C}_{14} \\
J_2(\beta) & Y_2(\beta) & I_2(\beta) & K_2(\beta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
J_3(\beta) & Y_3(\beta) & -I_3(\beta) & K_3(\beta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
J_2(\delta_1) & Y_2(\delta_1) & I_2(\delta_1) & K_2(\delta_1) & -J_2(\delta_1) & -Y_2(\delta_1) & -I_2(\delta_1) & -K_2(\delta_1) & 0 & 0 & 0 & 0 & 0 & 0 \\
J_3(\delta_1) & Y_3(\delta_1) & -I_3(\delta_1) & K_3(\delta_1) & -J_3(\delta_1) & -Y_3(\delta_1) & I_3(\delta_1) & -K_3(\delta_1) & 0 & 0 & 0 & 0 & 0 & 0 \\
J_4(\delta_1) & Y_4(\delta_1) & I_4(\delta_1) & K_4(\delta_1) & -J_4(\delta_1) & -Y_4(\delta_1) & -I_4(\delta_1) & -K_4(\delta_1) & 0 & 0 & 0 & 0 & 0 & 0 \\
\Delta_{11} & \Delta_{21} & \Delta_{31} & \Delta_{41} & -\Delta_{51} & -\Delta_{61} & -\Delta_{71} & -\Delta_{81} & 0 & 0 & 0 & 0 & 0 & 0 \\
\xi_1^{-1}J_2(\delta_1) & \xi_1^{-1}Y_2(\delta_1) & \xi_1^{-1}I_2(\delta_1) & \xi_1^{-1}K_2(\delta_1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+\gamma_1^2 & 0 \\
0 & 0 & 0 & 0 & J_2(\delta_2) & Y_2(\delta_2) & I_2(\delta_2) & K_2(\delta_2) & -J_2(\delta_2) & -Y_2(\delta_2) & -I_2(\delta_2) & -K_2(\delta_2) & 0 & 0 \\
0 & 0 & 0 & 0 & J_3(\delta_2) & Y_3(\delta_2) & -I_3(\delta_2) & K_3(\delta_2) & -J_3(\delta_2) & -Y_3(\delta_2) & I_3(\delta_2) & -K_3(\delta_2) & 0 & 0 \\
0 & 0 & 0 & 0 & J_4(\delta_2) & Y_4(\delta_2) & I_4(\delta_2) & K_4(\delta_2) & -J_4(\delta_2) & -Y_4(\delta_2) & -I_4(\delta_2) & -K_4(\delta_2) & 0 & 0 \\
0 & 0 & 0 & 0 & \Delta_{12} & \Delta_{22} & \Delta_{32} & \Delta_{42} & -\Delta_{52} & -\Delta_{62} & -\Delta_{72} & -\Delta_{82} & 0 & 0 \\
0 & 0 & 0 & 0 & \xi_2^{-1}J_2(\delta_2) & \xi_2^{-1}Y_2(\delta_2) & \xi_2^{-1}I_2(\delta_2) & \xi_2^{-1}J_2(\delta_2) & 0 & 0 & 0 & 0 & 0 & -1+\gamma_1^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_4(\beta\sqrt{\alpha}) & Y_4(\beta\sqrt{\alpha}) & I_4(\beta\sqrt{\alpha}) & K_4(\beta\sqrt{\alpha}) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & 0 & 0
\end{array}\right]
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14
\end{array}
\qquad \text{(A2)}
$$

where

$$\varepsilon_1 = [8J_4(\beta\sqrt{\alpha}) - \beta\alpha^{1/2}J_5(\beta\sqrt{\alpha})], \quad \varepsilon_2 = [8Y_4(\beta\sqrt{\alpha}) - \beta\alpha^{1/2}Y_5(\beta\sqrt{\alpha})],$$

$$\varepsilon_3 = [8I_4(\beta\sqrt{\alpha}) + \beta\alpha^{1/2}I_5(\beta\sqrt{\alpha})], \quad \varepsilon_4 = [8K_4(\beta\sqrt{\alpha}) - \beta\alpha^{1/2}K_5(\beta\sqrt{\alpha})],$$

$$\beta = \frac{2L\Omega}{(\alpha - 1)},$$

$$\Delta_{11} = 8\beta^2 J_4(\beta\sqrt{\xi_1}) - \beta^3\xi_1^{1/2}J_5(\beta\sqrt{\xi_1}) + 8\theta_1\xi_1^{-2}J_2(\beta\sqrt{\xi_1}), \quad \delta_1 = \beta\sqrt{\xi_1},$$

$$\Delta_{21} = 8\beta^2 Y_4(\beta\sqrt{\xi_1}) - \beta^3\xi_1^{1/2}Y_5(\beta\sqrt{\xi_1}) + 8\theta_1\xi_1^{-2}Y_2(\beta\sqrt{\xi_1}),$$

$$\Delta_{31} = 8\beta^2 I_4(\beta\sqrt{\xi_1}) + \beta^3\xi_1^{1/2}I_5(\beta\sqrt{\xi_1}) + 8\theta_1\xi_1^{-2}I_2(\beta\sqrt{\xi_1}),$$

$$\Delta_{41} = 8\beta^2 K_4(\beta\sqrt{\xi_1}) - \beta^3\xi_1^{1/2}K_5(\beta\sqrt{\xi_1}) + 8\theta_1\xi_1^{-2}K_2(\beta\sqrt{\xi_1}),$$

$$\Delta_{51} = 8\beta^2 J_4(\beta\sqrt{\xi_1}) - \beta^3\xi_1^{1/2}J_5(\beta\sqrt{\xi_1}),$$

$$\Delta_{61} = 8\beta^2 Y_4(\beta\sqrt{\xi_1}) - \beta^3\xi_1^{1/2}Y_5(\beta\sqrt{\xi_1}),$$

$$\Delta_{71} = 8\beta^2 I_4(\beta\sqrt{\xi_1}) + \beta^3\xi_1^{1/2}I_5(\beta\sqrt{\xi_1}),$$

$$\Delta_{81} = 8\beta^2 K_4(\beta\sqrt{\xi_1}) - \beta^3\xi_1^{1/2}K_5(\beta\sqrt{\xi_1}),$$

$$\Delta_{12} = 8\beta^2 J_4(\beta\sqrt{\xi_2}) - \beta^3\xi_2^{1/2}J_5(\beta\sqrt{\xi_2}) + 8\theta_1\xi_2^{-2}J_2(\beta\sqrt{\xi_2}),$$

$$\delta_2 = \beta\sqrt{\xi_2},$$

$$\Delta_{22} = 8\beta^2 Y_4(\beta\sqrt{\xi_2}) - \beta^3\xi_2^{1/2}Y_5(\beta\sqrt{\xi_2}) + 8\theta_2\xi_2^{-2}Y_2(\beta\sqrt{\xi_2}),$$

$$\Delta_{32} = 8\beta^2 I_4(\beta\sqrt{\xi_2}) + \beta^3\xi_2^{1/2}I_5(\beta\sqrt{\xi_2}) + 8\theta_2\xi_2^{-2}I_2(\beta\sqrt{\xi_2}),$$

$$\Delta_{42} = 8\beta^2 K_4(\beta\sqrt{\xi_2}) - \beta^3\xi_2^{1/2}K_5(\beta\sqrt{\xi_2}) + 8\theta_2\xi_2^{-2}K_2(\beta\sqrt{\xi_2}),$$

$$\Delta_{52} = 8\beta^2 J_4(\beta\sqrt{\xi_2}) - \beta^3\xi_2^{1/2}J_5(\beta\sqrt{\xi_2}),$$

$$\Delta_{62} = 8\beta^2 Y_4(\beta\sqrt{\xi_2}) - \beta^3\xi_2^{1/2}Y_5(\beta\sqrt{\xi_2}),$$

$$\Delta_{72} = 8\beta^2 I_4(\beta\sqrt{\xi_2}) + \beta^3\xi_2^{1/2}I_5(\beta\sqrt{\xi_2}),$$

$$\Delta_{82} = 8\beta^2 K_4(\beta\sqrt{\xi_2}) - \beta^3 \xi_2^{1/2} K_5(\beta\sqrt{\xi_2}),$$

$$\theta_1 = \left\{ \frac{m_1^*[\frac{1}{3}(\alpha-1)^2 + (\alpha-1) + 1](\Omega L)^4}{1 - m_1^*[\frac{1}{3}(\alpha-1)^2 + (\alpha-1) + 1](\Omega L)^4/k_1^*} \frac{1}{(\alpha-1)^3} \right\},$$

$$\gamma_1^2 = \frac{m_1^*[\frac{1}{3}(\alpha-1)^2 + (\alpha-1) + 1]}{k_1^*} \cdot (\Omega L)^4,$$

$$\theta_2 = \left\{ \frac{m_2^*[\frac{1}{3}(\alpha-1)^2 + (\alpha-1) + 1](\Omega L)^4}{1 - m_2^*[\frac{1}{3}(\alpha-1)^2 + (\alpha-1) + 1](\Omega L)^4/k_2^*} \frac{1}{(\alpha-1)^3} \right\},$$

$$\gamma_2^2 = \frac{m_2^*[\frac{1}{3}(\alpha-1)^2 + (\alpha-1) + 1]}{k_2^*} \cdot (\Omega L)^4$$

APPENDIX B

The coefficient matrices for the left end of the beam, $[B_L]$, and those for the right end of the beam, $[B_R]$, with the FC, SC, CS, CC and SS boundary conditions were given below.

(1) Free–clamped beam

$$[B_R] = \overset{\qquad\quad 1 \qquad\qquad\qquad 2 \qquad\qquad\qquad 3 \qquad\qquad\qquad 4}{\begin{bmatrix} J_4(\beta) & Y_4(\beta) & I_4(\beta) & K_4(\beta) \\ 8J_4(\beta) - \beta J_5(\beta) & 8Y_4(\beta) - \beta Y_5(\beta) & 8I_4(\beta) + \beta I_5(\beta) & 8K_4(\beta) - \beta K_5(\beta) \end{bmatrix}} \begin{matrix} 1 \\ 2 \end{matrix}, \quad \text{(B1)}$$

$$[B_L] = \overset{\quad 4n+1 \qquad 4n+2 \qquad 4n+3 \qquad 4n+4}{\begin{bmatrix} J_2(\beta\sqrt{\alpha}) & Y_2(\beta\sqrt{\alpha}) & I_2(\beta\sqrt{\alpha}) & K_2(\beta\sqrt{\alpha}) \\ J_3(\beta\sqrt{\alpha}) & Y_3(\beta\sqrt{\alpha}) & -I_3(\beta\sqrt{\alpha}) & K_3(\beta\sqrt{\alpha}) \end{bmatrix}} \begin{matrix} p-1 \\ p \end{matrix}, \quad \text{(B2)}$$

where

$p = 5n + 4.$

(2) Simply supported–clamped beam

$$[B_R] = \overset{\quad 1 \qquad\quad 2 \qquad\quad 3 \qquad\quad 4}{\begin{bmatrix} J_2(\beta) & Y_2(\beta) & I_2(\beta) & K_2(\beta) \\ J_4(\beta) & Y_4(\beta) & I_4(\beta) & K_4(\beta) \end{bmatrix}} \begin{matrix} 1 \\ 2 \end{matrix}, \quad \text{(B3)}$$

$$[B_L] = \overset{\quad 4n+1 \qquad 4n+2 \qquad 4n+3 \qquad 4n+4}{\begin{bmatrix} J_2(\beta\sqrt{\alpha}) & Y_2(\beta\sqrt{\alpha}) & I_2(\beta\sqrt{\alpha}) & K_2(\beta\sqrt{\alpha}) \\ J_3(\beta\sqrt{\alpha}) & Y_3(\beta\sqrt{\alpha}) & -I_3(\beta\sqrt{\alpha}) & K_3(\beta\sqrt{\alpha}) \end{bmatrix}} \begin{matrix} p-1 \\ p \end{matrix}. \quad \text{(B4)}$$

(3) Clamped–simply supported

$$[B_R] = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} J_2(\beta) & Y_2(\beta) & I_2(\beta) & K_2(\beta) \\ J_3(\beta) & Y_3(\beta) & -I_3(\beta) & K_3(\beta) \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} \end{array}, \tag{B5}$$

$$[B_L] = \begin{array}{c} \begin{matrix} 4n+1 & 4n+2 & 4n+3 & 4n+4 \end{matrix} \\ \begin{bmatrix} J_2(\beta\sqrt{\alpha}) & Y_2(\beta\sqrt{\alpha}) & I_2(\beta\sqrt{\alpha}) & K_2(\beta\sqrt{\alpha}) \\ J_4(\beta\sqrt{\alpha}) & Y_4(\beta\sqrt{\alpha}) & I_4(\beta\sqrt{\alpha}) & K_4(\beta\sqrt{\alpha}) \end{bmatrix} \begin{matrix} p-1 \\ p \end{matrix} \end{array}. \tag{B6}$$

(4) Clamped–clamped

$$[B_R] = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} J_2(\beta) & Y_2(\beta) & I_2(\beta) & K_2(\beta) \\ J_3(\beta) & Y_3(\beta) & -I_3(\beta) & K_3(\beta) \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} \end{array}, \tag{B7}$$

$$[B_L] = \begin{array}{c} \begin{matrix} 4n+1 & 4n+2 & 4n+3 & 4n+4 \end{matrix} \\ \begin{bmatrix} J_2(\beta\sqrt{\alpha}) & Y_2(\beta\sqrt{\alpha}) & I_2(\beta\sqrt{\alpha}) & K_2(\beta\sqrt{\alpha}) \\ J_3(\beta\sqrt{\alpha}) & Y_3(\beta\sqrt{\alpha}) & -I_3(\beta\sqrt{\alpha}) & K_3(\beta\sqrt{\alpha}) \end{bmatrix} \begin{matrix} p-1 \\ p \end{matrix} \end{array}. \tag{B8}$$

(5) Simply supported–simply supported beam

$$[B_R] = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} J_2(\beta) & Y_2(\beta) & I_2(\beta) & K_2(\beta) \\ J_4(\beta) & Y_4(\beta) & I_4(\beta) & K_4(\beta) \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} \end{array}, \tag{B9}$$

$$[B_L] = \begin{array}{c} \begin{matrix} 4n+1 & 4n+2 & 4n+3 & 4n+4 \end{matrix} \\ \begin{bmatrix} J_2(\beta\sqrt{\alpha}) & Y_2(\beta\sqrt{\alpha}) & I_2(\beta\sqrt{\alpha}) & K_2(\beta\sqrt{\alpha}) \\ J_4(\beta\sqrt{\alpha}) & Y_4(\beta\sqrt{\alpha}) & I_4(\beta\sqrt{\alpha}) & K_4(\beta\sqrt{\alpha}) \end{bmatrix} \begin{matrix} p-1 \\ p \end{matrix} \end{array}. \tag{B10}$$